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VARIATIONAL PRINCIPLES OF NON-LINEAR THEORY OF BRITTLE FRACTURE*

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An energy criterion of equilibrium of a non-linearly elastic body with a crack is formulated. Equations of statics and conditions which must hold at the outer boundary of the body, at its surface and at the slit edge, are derived. An evolutionary variational inequality is postulated, from which the formulation of the dynamic problem of the motion of a body with an expanding crack follow.

1. Forumulation of the problem. Let us consider an elastic solid which has a defect when in its natural state. The defect can be modelled by a displacement discontinuity surface, which will be called, from now on, the crack. Let this crack be situated on a smooth, two-dimensional surface Ω, with a smooth boundary $\partial\Omega$. We take the natural configuration of the body occupying the region $V_\Omega = V \setminus (\Omega \cup \partial\Omega)$ of three-dimensional Euclidean space as the reference configuration, and denote the Cartesian coordinates of the particles of the body in this configuration by X_a, $a = 1, 2, 3$. In the deformed state the Cartesian coordinates of the particles will be given by the formulas

$$x_i = x_i(X_1, X_2, X_3), \quad i = 1, 2, 3$$

The coordinates x_i fill the volume v of the current configuration. If the deformed body with a crack is in a state of equilibrium, the functions $x_i(X_a)$ will map in 1:1 correspondence with a positive Jacobian. When X_a pass through Ω, the functions x_i become discontinuous. The traces $x_i(X_a)$ on both sides of Ω describe the surfaces of the crack in the deformed state (Fig.1).

The first problem consists of establishing the criterion of equilibrium of the configuration $x_i(X_a)$. With this purpose in mind, we shall formulate the following variational principle: in order for the defromed body with a crack to remain in equilibrium, it is necessary and sufficient that the variation in the energy of the body taken in a specified configuration be greater than, or equal to zero for all admissible configurations. We shall call a virtual configuration of the body admissible, if its displacement discontinuity surface contains Ω, or if it coincides with it.

If the body has no crack, the criterion of equilibrium in the class of all continuous configurations reduces to the well-known principle of stationarity of the energy of a non-linearly elastic body /1-3/. The generality of the energy criterion of equilibrium was satisfactorily demonstrated for other mechanical systems by Gibbs /4/. The papers by Griffiths'

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/5, 6/ provide the source for a generalization of this criterion to the mechanics of the brittle fracture of a body with a crack. In order to find the critical length of the crack (in the plane problem) Griffiths differentiated the total energy of the body which included the surface energy of the crack, along the crack, and equated it to zero. This idea was further developed in numerous publications (e.g. /7-12/).

In the present paper the criterion is used to find the equations of statics and the boundary conditions in the problem of the equilibrium of a geometrically and physically non-linear body with a crack. We note that imposing a restriction on the admissible configurations in the energy criterion prevents the body returning to the state with a "recovered" crack. Therefore, the theory is distinctly non-holonomic and irreversible. As a result, the condition of equilibrium (or of non-propagation) of the crack consists of the requirement that the modulus of the transverse energy flux arriving at the crack edge should be less than, or equal to double the surface energy density. When the linearization is carried out, the condition reduces to the well-known Cherepanov condition /7, 8/.

If the condition of equilibrium does not hold for even a single configuration, the crack will become a propagating crack. The problem of the motion of a body with a crack, even in the material description, is a problem with a varying inner boundary. Here we have a typical mechanical system, with constraints being released. The most suitable method of describing such systems is that of the evolutionary variational inequality /13, 14/. In the case of a body with a crack, it is best to begin the study of variational inequality by analysing the equation of energy balance. Such an analysis shows why the kinetic energy flux must complement the work of the D'Alembert's inertia force in the evolutionary variational inequality of the theory of fracture. While postulating the evolutionary variational inequality, we must also demand that when the possible variations are replaced by the real velocities, the inequality will become the equation of energy balance /2, 3/. Only then can we find the complete system of relations necessary to formulate a dynamic problem.

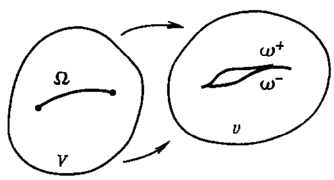

Fig.1

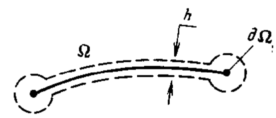

Fig.2

2. *Criterion of equilibrium.* Thus, if we take the criterion of equilibrium formulated above as the starting point, the formulation of the boundary-value problem of the equilibrium of an elastic body with a crack will reduce to that of specifying its energy on all admissible configurations in which the surfaces of discontinuity differ from Ω. Let some arbitrary configuration $x_i(X_a)$ have the surface of discontinuity Σ. By analogy with the Griffiths theory we postulate the following expression for the functional of the energy of the body:

$$E = \int_{V_\Sigma} U(x_{i,a}, K_B)\, dX + \int_{\Sigma} 2\gamma\, dA + \int_{V_\Sigma} \rho_0 \Phi(x_i)\, dX - \int_{\partial V_T} T_i x_i\, dA \tag{2.1}$$

$$V_\Sigma = V \setminus (\Sigma \cup \partial\Sigma) \quad \partial V = \partial V_x \cup \partial V_T$$

Here dX and dA are the volume and area elements respectively, ρ_0 is the mass density of the material in its normal state, $U(x_{i,a}, K_B)$ is the volume internal energy density and γ is the surface energy density. The tensor $x_{i,a} = \partial x_i / \partial X_a$ corresponds to the locality gradient (distortion), and $K_B(X_a)$ $(B = 1, \ldots, N)$ to the characteristics of the material (of the modulus of elasticity type). The mass force potential is denoted by $\Phi(x_i)$, and T_i is the "dead" load acting on the segment ∂V_T of the outer boundary of the body. The position of the particles on the remaining part of the outer boundary ∂V_x is assumed given: $x_i = r_i(X_a)$. Here and henceforth the lower case Latin indices assume the values 1, 2, 3 a coma preceding the index denotes a partial derivative in X_a, and repeated indices denote summation.

According to the energy criterion, the configuration of the body $x_i(X_a)$ with discontinuity surface Ω will be in equilibrium if for all admissible configurations $y_i = y_i(X_a, \varepsilon)$ with discontinuity surfaces $\Omega^\varepsilon \supset \Omega$ satisfying the constraints $y_i(X_a, 0) = x_i(X_a)$ and $y_i(X_a, \varepsilon) = r_i(X_a)$ at $X_a \in \partial V_x$, the following inequality holds:

$$\delta E = dE\,[y_i(X_a, \varepsilon)] / d\varepsilon\,|_{\varepsilon=0} \geqslant 0 \tag{2.2}$$

To obtain a corollary from (2.2) we must determine δE. Since the discontinuity surfaces in admissible configurations may differ from Ω, when $\varepsilon \neq 0$, it follows that in calculating δE it is convenient to introduce a 1:1 mapping of V onto V according to the rule $Y_a = Y_a(X_a, \varepsilon)$, such that the surface Ω will become Ω^ε and $Y(X_a, \varepsilon) = X_a$ when $\varepsilon = 0$ and when $X_a \in \partial V$ (the outer boundary is fixed with respect to the particles).

First we shall determine, separately, the variation of the internal energy

$$\delta U = \delta \int_{V_\Omega} U(y_{i,a'}, K_B) \det|Y_{a.b}|\, dX = \int_{V_\Omega} [T_{ai}\delta(y_{i,a'}) + (\partial U/\partial K_B) K_{B,a}\delta Y_a + U\delta Y_{a,a}]$$

Here and henceforth the symbol δ will be used to denote a partial derivative in ε when $\varepsilon = 0$, $X_a = \text{const}$. For example, $\delta Y_a = \partial Y_a(X_a, \varepsilon)/\partial\varepsilon\,|_{\varepsilon=0}$, $T_{ai} = \partial U/\partial x_{i,a}$ is the Piola-Kirchhoff tensor and $y_{i,a'} = \partial y_i/\partial Y_a$. It can be shown that $\delta(y_{i,a'}) = \delta y_{i,a} - x_{i,b}\,\delta Y_{b,a}$, $\delta y_i = \partial y_i(Y_a(X_a, \varepsilon)\, \varepsilon)/\partial\varepsilon$, therefore

$$\delta U = \int_{V_\Omega} [T_{ai}\delta y_{i,a} + \mu_{ab}\delta Y_{a,b} + (\partial U/\partial K_B) K_{B,a}\delta Y_a]\, dX \tag{2.3}$$

where $\mu_{ab} = -T_{bi}x_{ia} + U\delta_{ab}$ is the analogue of the chemical potential tensor in the theory of phase transfer /2, 4/. Since $x_i(X_a)$ and other functions become discontinuous on Ω and may have a singularities in the neighbourhood of $\partial\Omega$, we transform the integral (2.3) by first replacing the domain of integration V_Ω by V_h with internal boundary Ω_h separated from $\partial\Omega$ by a distance of the order of h (Fig.2).

Integrating the relation (2.3) by parts and letting h tend to zero, we obtain

$$\delta U = \int_{V_\Omega} [-T_{ai,a}\delta y_i + (-\mu_{ab,b} + (\partial U/\partial K_B) K_{B,a})\delta Y_a]\, dX + \int_\Omega [\{T_{ai}\delta y_i\} N_a + \{\mu_{ab}\} N_b \delta Y_a]\, dA - \int_{\partial\Omega} J_a \delta Y_a\, dS + \int_{\partial V_T} T_{ai}\delta y_i N_a\, dA \quad (\{f\} = f^- - f^+) \tag{2.4}$$

Here dS is the element of length, the plus and minus superscripts denote the limiting values of the corresponding magnitudes on both sides of Ω, N_a in the vector of the outer normal (it is directed, on Ω, towards the side corresponding to the plus sign). Finally, J is the vector of the energy flux arriving at the crack edge given by the formula

$$J_a = \lim_{|\Gamma|\to 0} \int_\Gamma \mu_{ab}\varkappa_b\, dS = \lim_{|\Gamma|\to 0} \int_\Gamma (-T_{bi}x_{i,a}\varkappa_b + U\varkappa_a)\, dS \tag{2.5}$$

where the closed contour Γ, in the plane transverse to $\partial\Omega$ embraces the point X_a on $\partial\Omega$ and contracts to it in the limit when the length of the contour $|\Gamma|$ tends to zero.

The integrals (2.5) are analogues of J, which represents the integral in the geometrically linear theory of fracture /7, 12/. We note that in deriving the relation (2.4) we assumed the following asymptotic behaviour of the stress field near the crack edge:

$$\lim_{|\Gamma|\to 0} \int_\Gamma T_{ai}\varkappa_a\, dS = 0$$

Such a property holds if the order of the singularity of T_{ai} near $\partial\Omega$ is less than unity.

The variations of the remaining terms of expression (2.1) have the form

$$\delta \int_{\Omega^\varepsilon} 2\gamma\, dA = -\int_\Omega 4\gamma H N_a \delta Y_a\, dA + \int_{\partial\Omega} 2\gamma\nu_a \delta Y_a\, dS \tag{2.6}$$

$$\delta \int_{\partial V_T} T_i y_i\, dA = \int_{\partial V_T} T_i \delta y_i\, dA, \quad \delta \int_{V\Omega^\varepsilon} \rho_0 \Phi(y_i)\, dX = \int_{V_\Omega} \rho_0 F_i(\delta y_i + x_{i,a}\delta Y_a)\, dX + \int_\Omega \rho_0 \{\Phi\} N_a \delta Y_a\, dA$$

where H is the mean curvature of the surface Ω, $F_i = -\partial\Phi/\partial x_i$ are the volume forces, ν_a is the vector of the normal to the crack edge $\partial\Omega$. By virtue of the above restrictions the virtual discontinuity surfaces Ω^ε always contain Ω and tend to it as $\varepsilon\to 0$. If the surfaces Ω^ε are smooth continuations of Ω, then ν_a will represent, at the same time, a vector tangent to Ω. In the general case this is optional, and ν_a just indicates a normal direction from $\partial\Omega$ to $\partial\Omega^\varepsilon$. Having combined relations (2.4) and (2.6), we arrive at the formula

$$\delta E = \int_{V_\Omega} [-(T_{ai,a}+\rho_0 F_i)\,\delta y_i + (-\mu_{ab,b} + (\partial U/\partial K_B)\,K_{B,a} + \rho_0 F_i x_{i,a})\,\delta Y_a]\,dX + \int_\Omega [\{T_{ai}\delta y_i\}N_a + (\{\mu_{ab}\}N_b - 4\gamma H N_a + \rho_0\{\Phi\}N_a)\,\delta Y_a]\,dA + \int_{\partial\Omega}(2\gamma\nu_a - J_a)\,\delta Y_a\,dS + \int_{\partial V_T}(T_{ai}N_a - T_i)\,\delta y_i\,dA \tag{2.7}$$

It is clear that δy_i and δY_a can take any prescribed values in Ω, just as δy_i can on ∂V_T. Therefore from (2.2) and (2.7) it follows that

$$T_{ai,a} + \rho_0 F_i = 0, \quad T_{ai} = \partial U/\partial x_{i,a} \tag{2.8}$$

$$-\mu_{ab,b} + (\partial U/\partial K_B)K_{B,a} + \rho_0 F_i x_{i,a} = 0 \tag{2.9}$$

$$x_i = r_i\,(X_a) \text{ on } \partial V_x, \quad T_{ai}N_a = T_i \text{ on } \partial V_T \tag{2.10}$$

We note however, that Eqs.(2.9) hold automatically by virtue of Eqs.(2.8).

In order to obtain the remaining relations, we shall analyse the constraints imposed on the variations δy_i, δY_a on Ω and $\partial\Omega$.

If the crack edges do not touch each other in the deformed state, then $\delta y_i^\pm$ can take any prescribed value.

Let us assume that the above statement is false. In this case we denote by Ω^+ and Ω^- the subregions of Ω, whose points are in contact after the deformation. If we superimpose on the surface Ω a two-dimensional curvilinear coordinate system, the contact condition will become

$$x_i^+(\eta_\alpha) = x_i^-(\theta_\alpha), \quad \eta_\alpha \in \Omega^+, \ \theta^-_\alpha \in \Omega^-, \ \alpha \in \{1, 2\}$$

We can show that the following inequality holds for the points η_α and θ_α:

$$[\delta y_i^+(\eta_\alpha) - \delta y_i^-(\theta_\alpha)]\, n_i \geqslant 0$$

where n_i is the general vector of the normal to the contacting surfaces directed towards the side with minus sign. In addition, we can show that the quantities $\delta y_i^\pm x_{i,\alpha}$ can take any value on Ω where $x_{i,\alpha} = dx_i/\partial\eta_\alpha$. In case of the slippage of crack edges, the frictional forces are disregarded.

The assumption that $\Omega^\varepsilon \supset \Omega$ implies the following constraints for the functions δY_a: $\delta Y_a N_a = 0$ on Ω, $\delta Y_a \nu_a \geqslant 0$, $\delta Y_a \pi_a = \delta Y_a \tau_a$ on $\partial\Omega$, where τ_a and π_a are the tangent vector and binormal to $\partial\Omega$, respectively.

Taking all the above constraints into account, we obtain the following boundary conditions from (2.2) and (2.7):

$$T^\pm_{ai}N_a = 0 \text{ on } \Omega\setminus\Omega^\pm, \quad T_{ai}N_a x_{i,a} = 0 \text{ on } \Omega^\pm \tag{2.11}$$

$$T^+_{ai}N_a n_i\sqrt{a}\,|_{\eta_\alpha} = T^-_{ai}N_a n_i\sqrt{a}\,|_{\theta_\alpha} = -p \leqslant 0 \quad \text{on } \Omega^\pm \tag{2.12}$$

$$\{\mu_{ab}\}N_b X_{a,\alpha} = 0 \text{ on } \Omega \tag{2.13}$$

$$|J_\alpha| = \sqrt{J_a J_a - J_3^2} \leqslant 2\gamma \tag{2.14}$$

$$(a = \det|a_{\alpha\beta}|, \quad a_{\alpha\beta} = X_{a,\alpha}X_{\alpha,\beta}, \quad X_{a,\alpha} = \partial X_a/\partial\eta_\alpha, \quad J_3 = J_a\tau_a)$$

Condition (2.13) holds identically by virtue of the other constraints.

Thus, relations (2.8), (2.10), (2.11), (2.12) and (2.14) represent, as a set, a complete system of equations and boundary conditions which must hold for any equilibrium configuration. We note that the condition of non-propagation of the crack (2.14) is separated from the remaining conditions. Therefore, when the problems are being solved in practice, we can first solve the system (2.8), (2.10), (2.11) and (2.12) so as to find the configuration $x_i\,(X_a)$ and stress field T_{ai}, and then use formula (2.5) to find J_a and confirm the condition (2.14).

3. ***Evolutionary variational inequality.*** If no configuration satisfies Eq.(2.8) and boundary conditions (2.10), (2.11), (2.12) and (2.14), the crack cannot be in equilibrium and it becomes a propagating crack. We denote the discontinuity surface of the body at the

instant t by Ω_t. Since the process of crack propagation is irreversible, it follows that we must impose the following constraint on Ω: $\Omega_{t'} \supset \Omega_t$ for $t' > t$. Thus even in the material description we have a problem with a changing inner boundary which is not specified in advance. The law of motion for a body with a crack has the form

$$x_i = x_i(X_a, t),\ X_a \in V_t = V \setminus (\Omega_t \cup \partial\Omega_t) \tag{3.1}$$

In order to obtain the equations of motion for $x_i(X_a, t)$, it is best to begin by analysing the energy balance

$$d(E + K)/dt = 0 \tag{3.2}$$

$$E = \int_{V_t} U(x_{i,a}, K_B)\, dX + \int_{\Omega_t} 2\gamma\, dA + \int_{V_t} \rho_0 \Phi(x_i)\, dX - \int_{\partial V_T} T_i x_i\, dA, \quad K = \frac{1}{2}\int_{V_t} \rho_0 x_i^{\cdot} x_i^{\cdot}\, dX \tag{3.3}$$

The functional E describes the potential energy of the body and surface energy of the crack, and K the kinetic energy of the body. Here $x_i^{\cdot} = \partial x_i(X_a, t)/\partial t$ is the velocity of the particles, and the remaining symbols are as before. Since E and K depend on t through the variables of the domain of integration, it follows that in order to fix these domains it is best to introduce the 1:1 mapping of V into V according to the rule $X_a' = X_a'(X_a, t')$, such, that the surface Ω_t passes into $\Omega_{t'}$; and $X_a'(X_a, t') = X_a$ when $t' = t$ and $X_a \in \partial V$. The functional E depends on t in exactly the same way as on ε, and hence in order to obtain $E^{\cdot}$ we only need to replace in formula (2.7) δE by $E^{\cdot}$, δy_i by $\delta_t x_i$ and δY_a by $X_a'^{\cdot}$, where the symbol δ_t denotes a derivative of a composite function of t with fixed X_a:

$$\delta_t x_i = \partial x_i(X_a'(X_a, t'), t')/\partial t'\,|_{t'=t}$$

This symbol is introduced here in order to stress the difference, for example, between $\delta_t x_i$ and $x_i^{\cdot}$. Let us now find $K^{\cdot}$:

$$K^{\cdot} = \frac{d}{dt'}\int_{V_t} \frac{1}{2}\rho_0 x_i^{\cdot} x_i^{\cdot} \det\left|\frac{\partial X_a'}{\partial X_b'}\right| dX\,|_{t'=t} = \int_{V_t} \delta_t\left(\frac{1}{2}\rho_0 x_i^{\cdot} x_i^{\cdot} \det\left|\frac{\partial X_a'}{\partial X_b'}\right|\right) dX = \int_{V_t}\left(\rho_0 x_i^{\cdot}\delta_t x_i^{\cdot} + \frac{1}{2}\rho_0 x_i^{\cdot} x_i^{\cdot} X_{a,a}'^{\cdot}\right) dX \tag{3.4}$$

After changing the coordinates the velocity of the particles takes the form

$$x_i^{\cdot} = \partial x_i(X_a', t')/\partial t'\,|_{t'=t}$$

It is clear that

$$x_i^{\cdot} = \delta_t x_i - x_{i,a} X_a'^{\cdot}, \quad \delta_t x_i^{\cdot} = x_i^{\cdot\cdot} + x_{i,a}^{\cdot} X_a'^{\cdot} \tag{3.5}$$

where $x_i^{\cdot\cdot} = \partial^2 x_i(X_a, t)/\partial t^2$ is the acceleration of the particles. Substituting the expression (3.5) for $\delta_t x_i^{\cdot}$ into formula (3.4) and integrating the last term by parts, we obtain

$$K^{\cdot} = \int_{V_t} \rho_0 x_i^{\cdot\cdot}(\delta_t x_i - x_{i,a} X_a'^{\cdot})\, dX + \int_{\partial\Omega_t} Q_a X_a'^{\cdot}\, dS, \quad Q_a = \lim_{|\Gamma_t|\to 0}\int_{\Gamma_t} \frac{1}{2}\rho_0 x_i^{\cdot} x_i^{\cdot} \varkappa_a\, dS \tag{3.6}$$

where Γ_t is a closed contour in the plane transverse to $\partial\Omega_t$, embracing the point X_a on $\partial\Omega_t$ and $\varkappa_a$ is the vector of the outer normal to Γ_t. The vector Q_a represents the kinetic energy flux arriving at the crack edge. Thus the law of conservation of energy for the body with a crack has the form

$$d(E + K)/dt = \int_{V_t} [(\rho_0 x_i^{\cdot\cdot} - T_{ai,a} - \rho_0 F_i)\delta_t x_i + (-\mu_{ab,b} + (\partial U/\partial K_B) K_{B,a} + \rho_0 F_i x_{i,a} - \rho_0 x_i^{\cdot\cdot} x_{i,a}) X_a'^{\cdot}]\, dX + \int_{\Omega_t} [\{T_{ai}\delta_t x_i\} N_a + \{\mu_{ab}\} N_b X_a'^{\cdot}]\, dA + \int_{\partial\Omega_t} (2\gamma\nu_a - I_a) X_a'^{\cdot}\, dS + \int_{\partial V_t} (T_{ai}N_a - T_i)\delta_t x_i\, dA \tag{3.7}$$

$$I_a = \lim_{|\Gamma_t|\to 0}\int_{\Gamma_t}\left[-T_{bi} x_{i,a}\varkappa_b + \left(U + \frac{1}{2}\rho_0 x_i^{\cdot} x_i^{\cdot}\right)\varkappa_a\right] dS \tag{3.8}$$

In (3.7) we have taken into account the fact that $N_a X_a^{\cdot\cdot} = 0$ on Ω_t (this property follows from the fact that $\Omega_{t'} \supset \Omega_t$ for $t' > t$).

Let us now consider the evolutionary variational inequality. To do this we introduce the class of admissible motions of the body with a crack $y_i(X_a, t, \varepsilon)$, with the discontinuity surface Ω_t^ε, satisfying the constraints $\Omega_t^\varepsilon \supset \Omega_t$ for all t. Since Ω_t^ε can differ from Ω_t, we shall construct, as before, the mapping from V into V according to the rule $Y_a = Y_a(X_a, t, \varepsilon)$ such that the surface Ω_t becomes Ω_t^ε and $Y_a(X_a, t, \varepsilon) = X_a$ when $\varepsilon = 0$ or $X_a \in \partial V$.

Next we shall formulate the principle of virtual work for the body with a crack as follows: in the real motion of the body with a crack the evolutionary variational inequality

$$\delta E + \int_{V_t} \rho_0 x_i^{\cdot\cdot} (\delta y_i - x_{i,a}\delta Y_a)\, dX - \int_{\partial\Omega_t} Q_a \delta Y_a\, dS \geqslant 0 \tag{3.9}$$

holds for all instants of time t and for all variations of the admissible motions δy_i, δY_a. Moreover, this principle will demand that the variational inequality (3.9) should become an equality expressing the law of conservation of energy, provided that δy_i and δY_a in (3.9) are replaced by $\delta_t x_i$ and $X_a^{\cdot\cdot}$. Here the symbol δ denotes the partial derivative in ε for fixed X_a, t, taken at $\varepsilon = 0$:

$$\delta y_i = \partial y_i (Y_a(X_a, t, \varepsilon), t, \varepsilon)/\partial\varepsilon\,|_{\varepsilon=0}$$
$$\delta Y_a = \partial Y_a (X_a, t, \varepsilon)/\partial\varepsilon\,|_{\varepsilon=0}$$

The functional E in (3.9) depends on the admissible functions $y_i(X_a, t, \varepsilon)$ with the discontinuity surfaces in the following manner:

$$E = \int_{V_t^\varepsilon} U(y_{i,a}, K_B)\, dX + \int_{\Omega_t^\varepsilon} 2\gamma\, dA + \int_{V_t^\varepsilon} \rho_0 \Phi\, dX - \int_{\partial V_T} T_i y_i\, dA$$
$$V_t^\varepsilon = V \setminus (\Omega_t^\varepsilon \cup \partial\Omega_t^\varepsilon)$$

The vector of the kinetic energy flux Q_a is given by (3.6). Comparing (3.6) and (3.7) with (3.9), we easily see why the variational inequality (3.9) contains, apart from the usual work done by inertia forces, the kinetic energy flux. However, the equation of energy balance appears to be only a heuristic concept and Eqs.(3.9) is regarded as a postulate in the dynamic theory of fracture.

Expanding the variation of energy E in inequality (3.9) we find, as before, that the sum of the integrals in (3.7) is non-negative when $\delta_t x_i$ is represented by δy_i and $X_a^{\cdot\cdot}$ by δY_a. The inequality, together with the constraints imposed on the functions δy_i and δY_a and listed above, leads to the relations

$$T_{ai,a} + \rho_0 F_i = \rho_0 x_i^{\cdot\cdot}, \quad T_{ai} = \partial U/\partial x_{i,a} \text{ in } V_t \tag{3.10}$$
$$x_i = r_i(X_a) \text{ on } \partial V_x, \quad T_{ai}N_a = T_i \text{ on } \partial V_T$$
$$T_{ai}^{\pm} N_a = 0 \text{ on } \Omega_t \setminus \Omega_t^{\pm}, \quad T_{ai}^{\pm} N_a x_{i,\alpha} = 0 \text{ on } \Omega_t^{\pm}$$
$$T_{ai}^{+} N_a n_i \sqrt{a}\,|_{\eta_\alpha} = T_{ai}^{-} N_a n_i \sqrt{a}\,|_{\theta_\alpha} = -p \leqslant 0 \quad \text{on} \quad \Omega_t^{\pm}$$
$$|I_\alpha| = \sqrt{I_a I_a - I_3^2} \leqslant 2\gamma \text{ on } \partial\Omega_t, \; I_3 = I_a \tau_a$$

where $\Omega_t^{\pm}$ are the prototypes of the contacting crack edges at the instant t, and I_a is the vector of the energy flux given by (3.8).

We shall now make use of the second part of the principle formulated above. Replacing δy_i, δY_a in the variational inequality (3.9) by $\delta_t x_i$, $X_a^{\cdot\cdot}$ and taking (3.10) into account, we obtain the law of conservation of energy in the form

$$\int_{\Omega} \{T_{ai}\delta_t x_i\} N_a\, dA + \int_{\partial\Omega_t} (2\gamma\nu_a - I_a) X_a^{\cdot\cdot}\, dS = 0 \tag{3.11}$$

Here $X_a^{\cdot\cdot}\nu_a$ determines the normal rate of propagation of the crack edge.

From (3.11) we obtain the additional relations

$$p > 0 \Rightarrow [x_i^{\cdot +}(\eta_\alpha) - x_i^{\cdot -}(\theta_\alpha)] n_i = 0 \tag{3.12}$$
$$p = 0 \Rightarrow [x_i^{\cdot +}(\eta_\alpha) - x_i^{\cdot -}(\theta_\alpha)] n_i \geqslant 0 \text{ on } \Omega_t^{\pm}$$
$$|I_\alpha| < 2\gamma \Rightarrow X_a^{\cdot\cdot}\nu_a = 0 \quad (\text{ no propagation })$$
$$|I_\alpha| = 2\gamma \Rightarrow X_a^{\cdot\cdot}\nu_a \geqslant 0, \quad 2\gamma\nu_a = I_a - I_3\tau_a$$

Relations (3.10) and (3.12) together form a complete system of equations and boundary conditions for determining the motion of a body with a crack.

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THE SELFSIMILAR DYNAMIC PROBLEM OF A HYDRAULIC CRACK WHEN ITS SIDES INTERACT WITH A CLEAVING GAS FLOW*

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A selfsimilar solution of the problem of the propagation of a hydraulic crack, taking into account the interaction of its edges with a cleaving gas flow, is obtained. The influence of this interaction on the stress intensity factor (SIF) and the dynamic flow characteristics is studied.

In the problems of cleavage of an elastic half-space by a rigid wedge, one of the factors influencing the SIF is the force of friction between the wedge and elastic medium /1/. When solving the quasistationary problems of the hydrofracture of a stratum, the firctional forces arising between the gas flow and the crack edges are taken into account only in the equation of motion of the flow. The shear stresses connected with the frictional forces are neglected when the equations of the theory of elasticity are solved /2/. The selfsimilar dynamic problems of the propagation of cracks cleaved by a gas flow were studied in this approximation in /5, 6/, using the method of functionally invariant solutions /3, 4/. When the cracks are cleaved by means of compressed gas at high velocities, as happens in the case of impulsive hydrofracture /7/ and in the problems of explosive fracture /8/, the shear stresses arising at the crack edges can become considerable.

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